

Quantum states experimentally achieving high-fidelity transmission over a spin chain

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A uniformly coupled double quantum Hamiltonian for a spin chain has recently been implemented experimentally. We propose a method for the determination of initial quantum states that will provide perfect or near-perfect state transmission for an arbitrary Hamiltonian including this one. By calculating the eigenvalues and eigenvectors of a unitary operator obtained from the free evolution plus an exchange operator, we find that the double quantum Hamiltonian spin chain will support a three-spin initial encoding that will transfer along the chain with remarkably high fidelity. The fidelity is also found to decrease very slowly with increasing chain length. In addition, we are able to explain previous results showing exceptional transfer using this method.

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Introduction.— Quantum information processing (QIP) often requires the transfer of known or unknown quantum states from one subspace to another within an information processing device. In recent years, the quantum spin chain has become a prime candidate for quantum communication purposes such as these [1–3]. In the simplest configuration, where the nearest neighbor couplings are considered to be equal, perfect state transmission is typically not possible between two single spin processors within a linear chain. In other words, there is typically a non-vanishing probability that the initial excitation amplitude can be found outside the receiving spin location [4] at any given time. In principle, however, perfect state transfer (PST) can be realized by properly engineering the couplings between neighboring sites [5]. High fidelity state transmissions can also be obtained using weakly coupled external qubits [6, 7], modifying only one or two couplings [8, 9], or by encoding the states using multiple spins [10–14]. In [13, 14], a class of states were found to transfer very well across long XY coupled spin chains. The existence of PST has also been established for a variety of interacting media, including, but not limited to, the spin chain model [15]. Recently, exact state swap through a spin ring has been investigated. It was shown that there is a straightforward approach to calculating the probability of the occurrence of an exact state swap [16].

The schemes developed in Ref. [15] prompted the following question. Given an arbitrary spin chain Hamiltonian, can we find initial states which can be used to enable high-fidelity state transmission? In this letter, we answer this question and show that for a uniformly coupled chain, there exists a particular state which reliably transfers quantum information over large distances. We use a multi-spin encoding scheme and find the existence

of a three-spin encoding which can provide reliable state transmission. In this case, the encoding and decoding processes can also be realized easily [13]. This report is therefore important from an experimental perspective due to the ease of implementation which is typically favorable.

The method for identifying high-fidelity states.— Consider a spin chain consisting of N sites which evolves according to some Hamiltonian H in a single excitation subspace. Suppose for the moment the initial state of our system is $|\Psi(0)\rangle = |\mathbf{1}\rangle = |1\rangle_A \otimes |101\dots 1\rangle \otimes |0\rangle_B$, where A and B denote separate processors. After the system evolves, the state at time t will be

$$|\Psi(t)\rangle = U(t) |\mathbf{1}\rangle = \exp(-iHt) |\Psi(0)\rangle, \quad (1)$$

where \hbar is taken to be 1 throughout. Suppose that at some time τ PST occurs, then

$$|\Psi(\tau)\rangle = U(\tau) |\mathbf{1}\rangle = |\mathbf{N}\rangle, \quad (2)$$

where $|\mathbf{N}\rangle = |0\rangle_A \otimes |101\dots 1\rangle \otimes |1\rangle_B$. We can use a permutation operator P_{AB} to swap all states in A and B, then the quantum information can be transferred from A to B. The permutation operator can be expressed as:

$$P_{AB} = \sum_{\alpha\beta} (|\beta_A\rangle \langle\alpha_A| \otimes |\alpha_B\rangle \langle\beta_B|), \quad (3)$$

where $\alpha, \beta = 1, 2, \dots, 2^k$ represent the standard basis for the k qubits located in processors A and B. Clearly $P_{AB}^\dagger = P_{AB}$ and $P_{AB}^2 = 1$. $|\alpha(\beta)_{A(B)}\rangle$ refers to a state $|\alpha(\beta)\rangle$ in processor A (B). From Eq. (2)

$$U(\tau) |\mathbf{1}\rangle = P_{AB} |\mathbf{1}\rangle, \quad (4)$$

Then

$$P_{AB} U(\tau) |\mathbf{1}\rangle = W(\tau) |\mathbf{1}\rangle = |\mathbf{1}\rangle. \quad (5)$$

We introduce the unitary operator $W(\tau) = P_{AB}U(\tau)$. From Eq. (5), if the state $|\mathbf{1}\rangle$ is an eigenvector of the operator W at time τ , PST occurs. The eigenvectors of W reveal information about the possibilities of a specific state transmission. The problem of solving Schrödinger's equation now becomes a standard eigen-problem of the operator W .

Since $W(\tau)$ is a unitary operator it has a complete set of orthonormal eigenvectors $\{|\Psi_m(0)\rangle\}_\tau$ corresponding to eigenvalues $\{E_m\}_\tau$,

$$W(\tau)|\Psi_m(0)\rangle = E_m|\Psi_m(0)\rangle. \quad (6)$$

This can also be written as

$$U(\tau)|\Psi_m(0)\rangle = E_m P_{AB}^\dagger |\Psi_m(0)\rangle, \quad (7)$$

where $U(\tau)|\Psi_m(0)\rangle$ is the wave function $|\Psi_m(\tau)\rangle$ of the system which was initially prepared in the eigenstate $|\Psi_m(0)\rangle$. If $|\Psi_m(0)\rangle$ is a product state

$$|\Psi_m(0)\rangle = |A\rangle \otimes |C\rangle, \quad (8)$$

with $|A\rangle$ describing the state of processor A and $|C\rangle$ describing the rest of the system, we can then obtain

$$|\Psi_m(\tau)\rangle = E_m P_{AB}^\dagger |A\rangle \otimes |C\rangle = E_m |B\rangle \otimes |C'\rangle. \quad (9)$$

For the single excitation subspace, if one of the eigenvectors $|\Psi_m(0)\rangle = |\mathbf{1}\rangle$ at time τ , PST occurs. If the eigenvectors are degenerate, an arbitrary linear superposition of these degenerate states is also suitable for PST. Suppose there are L degenerate eigenvectors $|\Psi_l(0)\rangle$ ($l = 1, 2, \dots, L$), which have common eigenvalues E_L . The state

$$|\Psi(0)\rangle = \sum_{l=1}^L C_l |\Psi_l(0)\rangle. \quad (10)$$

is an eigenvector of $W(\tau)$, where C_l is an arbitrary number. Our analysis describes a method for finding a state which can realize PST. (Note that these states are not all unique.) For a given Hamiltonian, if we initially prepare the state $|\Psi(0)\rangle$ as an eigenvector of the operator $W(\tau)$, then after time τ PST occurs.

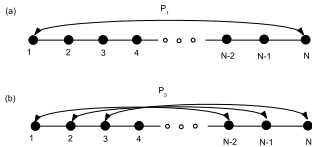


FIG. 1: Schematic of our quantum transmission protocol: (a) single-site encoding (b) three-site encoding.

For state transmission from one end spin 1 to another end spin N , the exchange operator is given by $P_1 = |\mathbf{1}\rangle\langle N| + |N\rangle\langle \mathbf{1}| + P_0$ which is shown in Fig. 1,

where $P_0 = \sum_{j \neq 1, N} |j\rangle\langle j|$. We now considered a particular Hamiltonian, but emphasize that our method can be used for any Hamiltonian not only this example.

An experimentally implementable Hamiltonian.— Now consider the recently implemented Hamiltonian called a double quantum (DQ) Hamiltonian [17]:

$$H = - \sum_{i=1}^{N-1} J_{i,i+1} (X_i X_{i+1} - Y_i Y_{i+1}). \quad (11)$$

where $J_{i,i+1}$ denotes the coupling between sites i and $i+1$. This nearest neighbor coupled one-dimensional spin chain can be experimentally implemented using solid-state nuclear magnetic resonance [17–19] in ^{19}F spins in a crystal of fluorapatite ((FAP-Ca₅(PO₄)₃F) [18, 19]. The system described by Eq. (11) will exhibit free evolution such that the evolution operator at time τ will be $U(\tau) = \exp[-i\tau H]$. We can diagonalize the Hamiltonian H such that $H_d = W^\dagger H W$ in the single excitation subspace. The evolution operator can therefore be expressed by $U(\tau) = W \exp[-i\tau H_d] W$ and the N eigenvectors of $W(\tau)$ can be obtained as a function of τ . Furthermore, we consider a natural configuration for a DQ Hamiltonian with open ends. The z -component of the total for the staggered spins is a conserved quantity, $[(\sum_{i \in \text{odd}} Z_i - \sum_{i \in \text{even}} Z_i), H] = 0$. For simplicity, we will only consider the single excitation subspace of the full Hilbert space. In this case the total number of flipped spins is one. The basis for this subspace will be denoted as $|\mathbf{j}\rangle$ which indicates that, after flipping, the even (odd) site spins all of the spins reside in the $|0\rangle$ ($|1\rangle$) state except for the spin at site j which is in the $|1\rangle$ ($|0\rangle$) state. For example, in a $N = 5$ site chain, the single excitation subspace will be spanned by $|\mathbf{1}\rangle = |1010\rangle, |\mathbf{2}\rangle = |00010\rangle, \dots$, etc. If we flip the even numbered states we find that the total up spin is actually one. We will use this description throughout this paper.

Example I: nonuniform couplings— We will consider several different coupling configurations with the potential for high-fidelity state transmission and the best results will be provided at the end of our analysis. First as an example, we consider two pre-engineered couplings: (1) weak couplings at both ends, where $J_{1,2} = J_{N-1,N} = J_0$ and $J_{i,i+1} = J$ elsewhere. (2) couplings termed PST, where $J_{i,i+1} = \sqrt{i(N-i)}$. It is already known that high fidelity ($F_{\max} \approx 1$) state transmission for the first configuration [6, 7] and perfect fidelity ($F_{\max} = 1$) for the second configuration can be gained in a spin system [5]. Here we will use these two kinds couplings to show the applicability of our methods.

For a five-spin system with weak couplings at both ends, we take $J_{1,2} = J_{4,5} = 0.1J$. J equals -1 elsewhere. The eigenvalues and eigenvectors of the operator $W(\tau)$ at an arbitrary time τ can be obtained numerically. We will consider those which span the single-excitation subspace. In Table I, we plot the results for $\tau = 31$. The first col-

| | 1⟩ | 2⟩ | 3⟩ | 4⟩ | 5⟩ | |
|----------|----------------------|--------------|---------------|---------------|---------------|---------------|
| 1 | (0.999,0.03) | 0.035 | -0.500 | 0.705 | -0.500 | 0.035 |
| 2 | (0.999,-0.03) | 0.035 | 0.500 | 0.705 | 0.500 | 0.035 |
| 3 | (1.000,0) | 0.398 | 0 | -0.066 | 0 | 0.914 |
| 4 | (1.000,0) | 0.999 | 0 | -0.050 | 0 | -0.003 |
| 5 | (-1.000,0) | 0 | 0.707 | 0 | -0.707 | 0 |

TABLE I: The complex eigenvalues and corresponding eigenvectors of the operator $W(\tau)$ at $\tau = 31$ in a spin chain where the weak coupling conditions $J_{1,2} = J_{4,5} = 0.1J$ are satisfied. We take $N=5$, $J = -1$.

umn labels the complex eigenvalues while the remaining columns are associated with the amplitudes of the states at the top of each column. The coefficients in columns 2-6 could be complex numbers but our results show that the imaginary part always equals zero, so we take them to be real throughout. The same meaning holds for Tables I, II, IV, and V. For example, at the line labeled with a 4 the eigenvalue is (1.000,0) and the eigenvector is $0.999|1\rangle - 0.050|3\rangle - 0.003|5\rangle$. The state $|1\rangle$ closely approximates this eigenvector and it can be written as the aforementioned product state. If we use the state $|1\rangle$ as the initial state of the whole system, then at time $\tau = 31$, the system will closely approximate the state $|N\rangle$.

We use the fidelity between the received state and the ideally transferred state, $F = \sqrt{\langle\Phi(0)|\rho(t)|\Phi(0)\rangle}$ as a measure of the quality of the transfer. Here $|\Phi(0)\rangle$ is a state at the receiving end which has the same form as the state initially prepared by the sender. $\rho(t)$ is the reduced density matrix of the receiver's spin at time t and is obtained by tracing over all but the receiver's sites. In Fig. 2 (a) we plot the fidelity versus time t for the weakly coupled chain. The initial state is $|1\rangle$ has the maximum fidelity, $F = 1$ at time $t = 31$.

| | 1) | 2) | 3) | 4) |
|------------------|-----------|-----------|-----------|-----------|
| 1 (0,1.000) | 0.707 | 0 | 0 | -0.707 |
| 2 (0.018,-1.000) | 0 | 0.707 | -0.707 | 0 |
| 3 (-0.009,1.000) | 0.612 | -0.354 | -0.354 | 0.612 |
| 4 (0.028,1.000) | 0.354 | 0.612 | 0.612 | 0.354 |

TABLE II: The complex eigenvalues and corresponding eigenvectors of the operator $W(\tau)$ at $\tau = 3.14$ in a $N=4$ site spin chain with couplings given by $J_{i,i+1} = \sqrt{i(N-i)}$.

Next we discuss PST. For the simple case of $N = 4$, the results at time $\tau = 3.14$ are listed in Table II. None of the eigenvectors can be written in the form of a product state $|\Psi_m(0)\rangle = |A\rangle \otimes |C\rangle$, but the eigenvalues of 1, 3, 4 are roughly degenerate. Consider the superposition

$$\sqrt{2}|\Psi_1(0)\rangle + \sqrt{\frac{3}{8}}|\Psi_3(0)\rangle + \sqrt{\frac{1}{8}}|\Psi_4(0)\rangle = |1\rangle, \quad (12)$$

where

$$\begin{aligned} |\Psi_1(0)\rangle &= \sqrt{\frac{1}{2}}(|1\rangle - |4\rangle), \\ |\Psi_3(0)\rangle &= \sqrt{\frac{3}{8}}(|1\rangle + |4\rangle) - \sqrt{\frac{1}{8}}(|2\rangle + |3\rangle), \\ |\Psi_4(0)\rangle &= \sqrt{\frac{1}{8}}(|1\rangle + |4\rangle) + \sqrt{\frac{3}{8}}(|2\rangle + |3\rangle). \end{aligned} \quad (13)$$

The state $|1\rangle$ at site 1 can be transferred exactly to site 4 at time $\tau = 3.14$. In Fig. 2(b) we plot the time evolution of the fidelity when transferring a state $|1\rangle$ from site 1 to 4. We also see that at time $\tau = 3.14$ the fidelity is nearly 1. These examples illustrate the validity and practicality of our method while providing a general method to obtain the results.

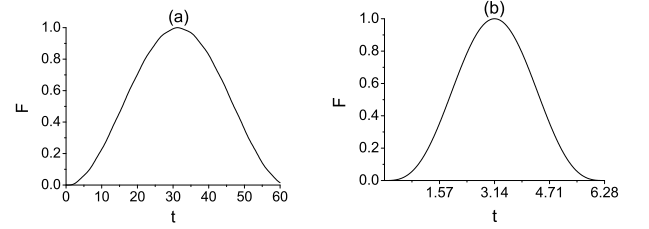


FIG. 2: The fidelity as a function of time for (a) $N = 5$, weak coupling conditions for $J_{1,2} = J_{4,5} = 0.1J$ (b) $N = 4$, channel coupling conditions $J_{i,i+1} = \sqrt{i(N-i)}$. See text for more details.

Example II: uniform couplings. —Consider the most natural configuration for a spin chain; a uniformly coupled spin chain. We take the ferromagnetic coupling $J_{i,i+1} = J = -1$. Note that PST is typically unattainable in these systems using single-spin encodings [5, 10]. We will consider both single spin encodings as well as multi-spin encodings. For single-spin encodings, our calculations confirm that PST cannot occur in this model as is known [16]. In table III we list the the maximal p_m and the corresponding τ for a $N = 7$ uniform chain for different eigenvectors. Here p_m is defined as the overlap between the eigenvector $|\Psi_m(0)\rangle$ and the initial state $|1\rangle$, i.e., $p_m = |\langle\Psi_m(0)|1\rangle|$.

| $ \Psi_m\rangle$ | $ \Psi_1\rangle$ | $ \Psi_2\rangle$ | $ \Psi_3\rangle$ | $ \Psi_4\rangle$ | $ \Psi_5\rangle$ | $ \Psi_6\rangle$ | $ \Psi_7\rangle$ |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| τ | 31.1 | 8 | 31 | 18.6 | 35.6 | 30.6 | 8.9 |
| p_m | 0.7071 | 0.6295 | 0.7062 | 0.6402 | 0.7068 | 0.6842 | 0.7071 |

TABLE III: The maximal p_m and corresponding values of τ . $N=7$, the maximum values are found in a time interval [5, 40].

Now we will examine multi-spin encoding schemes. As an example, we first consider a three-spin encoding. Specifically, suppose we wish to transfer a state of the form $|\Psi(0)\rangle = (\alpha|100\rangle + \beta|010\rangle + \gamma|001\rangle)_A \otimes |00\dots 0\rangle \otimes |000\rangle_B$. As shown in Fig. 1(b), we intend to transfer the

state of the first three spins to the opposite end. The results for a $N = 6$ site chain are given in Table IV for time $\tau = 4.0$. The eigenvalues of 1 and 2 are roughly degenerate and the approximate relation

$$|\Psi_1(0)\rangle + |\Psi_2(0)\rangle = -|\mathbf{1}\rangle + |\mathbf{3}\rangle \quad (14)$$

can be written in the form of a product state $(-|\mathbf{110}\rangle + |\mathbf{011}\rangle)_A \otimes |\mathbf{10...1}\rangle$. The state $(-|\mathbf{110}\rangle + |\mathbf{011}\rangle)/\sqrt{2}$ is therefore suitable for transmission. We have also checked the case where $N = 7$ and find that at time $\tau = 28.8$ the above states can be obtained again.

| | $ \mathbf{1}\rangle$ | $ \mathbf{2}\rangle$ | $ \mathbf{3}\rangle$ | $ \mathbf{4}\rangle$ | $ \mathbf{5}\rangle$ | $ \mathbf{6}\rangle$ |
|-------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| 1 (0.117,-0.993) | -0.493 | 0.005 | 0.500 | 0.500 | 0.005 | -0.500 |
| 2 (-0.117,-0.993) | -0.493 | -0.005 | 0.500 | -0.500 | 0.005 | 0.500 |
| 3 (0.252,0.968) | -0.275 | 0.590 | -0.275 | -0.275 | 0.590 | -0.275 |
| 4 (-0.252,0.968) | 0.275 | 0.590 | 0.275 | -0.275 | -0.590 | -0.275 |
| 5 (0.544,0.839) | 0.421 | 0.379 | 0.405 | 0.405 | 0.379 | 0.421 |
| 6 (-0.54,0.839) | -0.421 | 0.379 | -0.405 | 0.405 | -0.379 | 0.421 |

TABLE IV: The eigenvalues and corresponding eigenvectors of the operator $W(\tau)$ at $\tau = 4.0$ using a 3 spin encoding. Here $N=6$.

| | $ \mathbf{1}\rangle$ | $ \mathbf{2}\rangle$ | $ \mathbf{3}\rangle$ | $ \mathbf{4}\rangle$ | $ \mathbf{5}\rangle$ |
|------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| 1 (0.999,-0.025) | -0.263 | -0.263 | 0.850 | -0.263 | -0.263 |
| 2 (-0.999,0.041) | 0.500 | -0.500 | 0.000 | -0.500 | 0.500 |
| 3 (-0.997,0.076) | 0.500 | -0.500 | 0.000 | 0.500 | -0.500 |
| 4 (0.997,0.076) | 0.500 | 0.500 | 0.000 | -0.500 | -0.500 |
| 5 (0.998,0.067) | 0.425 | 0.425 | 0.526 | 0.425 | 0.425 |

TABLE V: The eigenvalues and corresponding eigenvectors of the operator $W(\tau)$ at $\tau = 47.2$ using a 2-spin encoding. Here $N=5$.

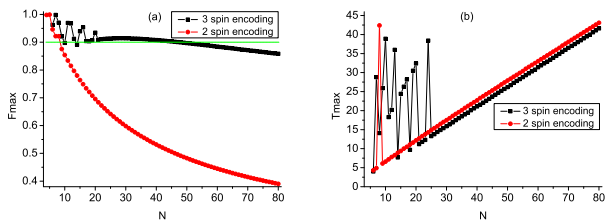


FIG. 3: (Color online.) Length dependence of the maximum fidelity achievable F_{max} and the associated arrival times T_{max} for the state (a) $(-|\mathbf{110}\rangle + |\mathbf{011}\rangle)/\sqrt{2}$ and (b) $(|\mathbf{11}\rangle - |\mathbf{00}\rangle)/\sqrt{2}$. The time is searched within the interval $[0, 50]$.

Table V lists the results corresponding to a $N = 5$ chain for the two-spin encoding. At $\tau = 47.2$, the relation $|\Psi_2(0)\rangle + |\Psi_3(0)\rangle = |\mathbf{2}\rangle - |\mathbf{1}\rangle$ approximately holds which can also be written in the form of the product state in Eq. (8).

We have found some states realizing high-fidelity state transmission, for small N . Now we will check to see if

the state $(-|\mathbf{11}\rangle + |\mathbf{00}\rangle)/\sqrt{2}$ and $(-|\mathbf{110}\rangle + |\mathbf{011}\rangle)/\sqrt{2}$ can be transferred with high fidelity across chains of arbitrary length N . In Fig. 3, we plot the maximum fidelity F_{max} and the associated arrival time T_{max} as a function of chain length N . The analytic expression with eigenvalues $E_m = -2J \cos[\pi m/(N+1)]$ and eigenvectors $|\Psi_m(0)\rangle = \sqrt{2/(N+1)} \sum_j \sin(q_m j) |\mathbf{j}\rangle$ are used. For practical implementation of our protocol, the maximum fidelity is found in the time $[0, 50]$. For the two-spin encoding, the high fidelity associated with short chain lengths cannot be achieved with increasing chain length. F_{max} quickly decreases with increasing N . However, this robustness can be observed even for long chains using the three-spin encoding. The fidelity is exceptionally large for a relatively long chain. Therefore, using this state, a high-fidelity state transfer can be gained. $F_{max} = 0.96$ for $N = 6$ at $t = 4.0$, $F_{max} = 1.00$ for $N = 7$ at $t = 28.8$ which agrees with our previous analysis [13, 14]. Note that we only consider two and three-spin encodings here. For encodings using more than three spins, we conjecture that for odd spin encodings some states can be found to possess high-fidelity transmission even over long chains. From Fig. 3 (b) we find that the arrival time T_{max} typically increases with increasing chain length N except for some deviation with small values of N . We also find that the T_{max} associated with the three-spin encoding is a little longer than in the two-spin encoding case for $N > 24$. This suggests that encodings using larger Hilbert spaces require longer waiting times for the maximum fidelity.

Conclusions.—In conclusion, we have introduced a method to find states which can be transmitted through spin channels with high fidelity. The method can be easily implemented numerically and can be applied to N -site encodings, with N arbitrary. Using our method we have provided examples for the DQ Hamiltonian which exhibit uniform and nonuniform exchange couplings. For the uniform chain, a 3-spin encoding $(-|\mathbf{110}\rangle + |\mathbf{011}\rangle)/\sqrt{2}$ was found to exhibit high fidelity state transmission. Using a simple similarity transformation [17], our results can be extended to the standard Heisenberg XY model. In this case we have provided an explanation for the appearance of the class of initial states which were previously discovered [13, 14]. These states are exceptional due to the fact that they use simple encodings and transfer extremely well. Our work therefore provides a new method, new results, and an explanation of previously known important results.

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